2021 CCL Winter Camp

Lyapunov Optimization Framework

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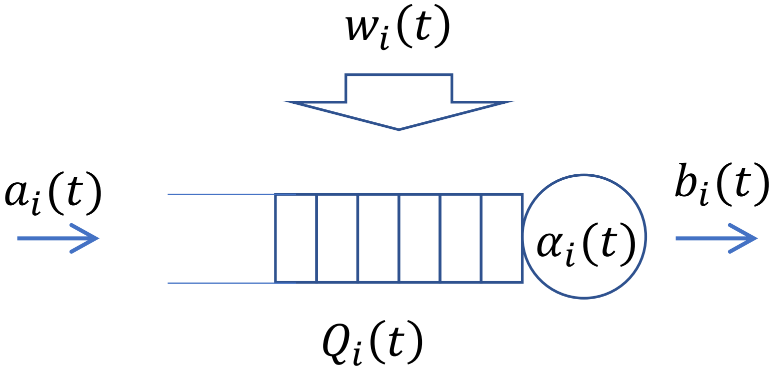
Use case: Optimizing the time-average energy subject to queue stability



$$Q_i(t+1) = \max\{Q_i(t) + a_i(t) - b_i(t), 0\}$$

$w_i(t)$: environmental variable at time t (i.i.d over slots)

$\alpha_i(t)$: control action at time t, determined based on $Q_i(t), a_i(t)$, and $w_i(t)$

$e_i(t)$: energy cost for action $\alpha_i(t)$, deterministic as $e_i(t) = f_e(\alpha_i(t), w_i(t))$

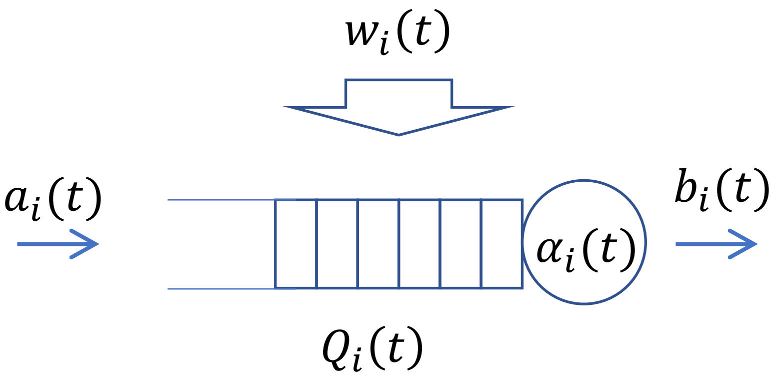
$b_i(t)$: output of action $\alpha_i(t)$, deterministic as $b(t) = f_b(\alpha_i(t))$

minimize: $\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} E[e_i(t)]$ — Time-average energy

subject to: $\lim_{t\to\infty} \frac{E[Q_i(t)]}{t} = 0$

Queues are mean-rate stable

Use case: Optimizing the time-average energy subject to queue stability



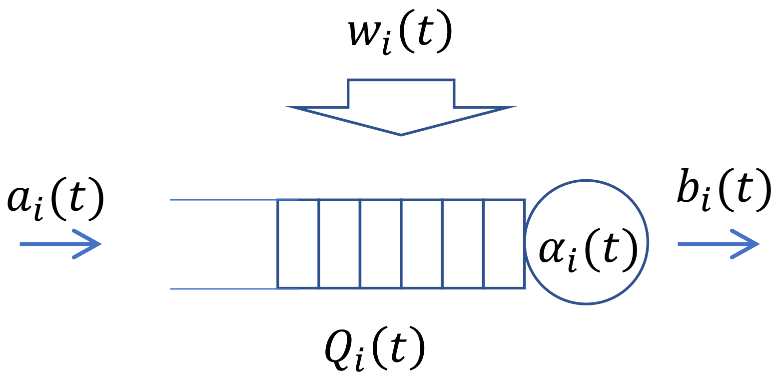
$$Q_i(t+1) = \max\{Q_i(t) + a_i(t) - b_i(t), 0\}$$

minimize: $E[e_i(t)]$

subject to: $\lim_{t \to \infty} \frac{E[Q_i(t)]}{t} = 0$

- **Greedy approach:**
  - Try to process $a_i(t)$ as much as possible in each time slot to achieve queue stability
    -> may **not energy-efficient**
    -> *Any alternative?*
  - Alternatively, limit $b_i(t)$ in each time slot to minimize the energy consumption
    -> *then, how to control queue stability?*
- **Classical optimization:**
  - Complete information is assumed, i.e., $a_i(t)$ & $w_i(t)$, $t = 0,1,2, \ldots$ are all available at the beginning,
  - Optimal solution could be obtained
  - But, how do we know the future for real-time control?
  - ***Online optimization*** *is indeed needed*

Lyapunov drift: An approach for dealing with queue stability for dynamic systems



$$Q_i(t+1) = \max\{Q_i(t) + a_i(t) - b_i(t), 0\}$$

Define the vector of queue backlogs at time t by

$$Q(t) = (Q_1(t), \ldots, Q_N(t))$$

(Suppose there are N queues in the network)

For each slot, define: $L(t) = \frac{1}{2}\sum_{i=1}^{N} Q_i(t)^2$

The Lyapunov drift: $\Delta L(t) = L(t+1) - L(t)$

$$\Delta L(t) \leqslant B(t) + \sum_{i=1}^{N} Q_i(t)(a_i(t) - b_i(t))$$

$$B(t) = \frac{1}{2}\sum_{i=1}^{N} (a_i(t) - b_i(t))^2$$

Suppose the second moments of arrivals and service in each queue are bounded:

$$\mathbb{E}[B(t)|Q(t)] \leqslant B$$

Lyapunov drift theorem

The conditional expected Lyapunov drift:

$$\mathbb{E}[\Delta L(t)|Q(t)] \leqslant B + \sum_{i=1}^{N} Q_i(t)\mathbb{E}[a_i(t) - b_i(t)|Q(t)]$$

Assume that the difference between arrivals and service satisfies the following property for some real $\varepsilon > 0$

$$\mathbb{E}[a_i(t) - b_i(t)|Q(t)] \leqslant -\varepsilon$$

Theorem (Lyapunov Drift):

$$\mathbb{E}[\Delta L(t)|Q(t)] \leqslant B - \varepsilon \sum_{i=1}^{N} Q_i(t).$$

The condition holds for all t and all Q_i

$$\frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{i=1}^{N}\mathbb{E}[Q_i(\tau)] \leqslant \frac{B}{\varepsilon} + \frac{\mathbb{E}[L(0)]}{\varepsilon t}.$$

Lyapunov drift-plus-penalty: minimize time averages with queue stability

The drift-plus-penalty function:

$$\Delta L(t) + V p(t) \qquad (1)$$

Drift function

Control parameter, non-negative

Penalty function, e.g., energy consumption

$$\text{E}[(1)] \leqslant B + \sum_{i=1}^{N} Q_i(t)\mathbb{E}[a_i(t) - b_i(t) | Q(t)] + Vp(t)$$

Theorem (Lyapunov Optimization):

$$\mathbb{E}[\Delta L(t) + Vp(t) | Q(t)] \leqslant B + Vp^* - \varepsilon \sum_{i=1}^{N} Q_i(t)$$

The condition holds for all t and all Q_i

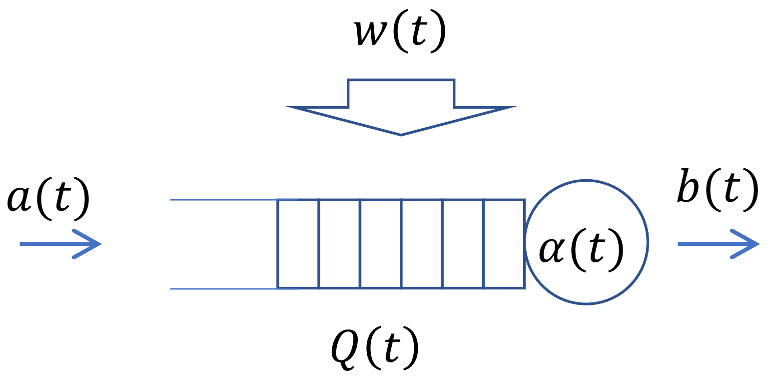
$$\frac{1}{t}\sum_{\tau=0}^{t-1}\mathbb{E}[p(\tau)] \leqslant p^* + \frac{B}{V} + \frac{\mathbb{E}[L(0)]}{Vt}$$

$$\frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{i=1}^{N}\mathbb{E}[Q_i(\tau)] \leqslant \frac{B + V(p^* - p_{\min})}{\varepsilon} + \frac{\mathbb{E}[L(0)]}{\varepsilon t}$$

p^*: the desired target for the time average of $p(t)$

$p(t) \geqslant p_{\min} \quad \forall t \in \{0, 1, 2, \ldots\}$

Lyapunov Optimization Algorithm



$$Q(t+1) = \max\{Q(t) + a(t) - b(t), 0\}$$

Energy consumption: $e(t) = f_e(b(t), w(t))$

minimize: $E[e(t)]$

subject to: $\lim_{t\to\infty} \frac{E[Q(t)]}{t} = 0$

$$L(t) = \frac{1}{2} Q^2(t)$$

$$Q^2(t+1) \le Q^2(t) + 2Q(t)\big(a(t) - b(t)\big) + \big(a(t) - b(t)\big)^2$$

$$\Delta L(t) = L(t+1) - L(t)$$

Assuming that $a(t)$ and $b(t)$ are bounded,

$$\Delta L(t) + Ve(t) \le B + \boxed{Q(t)\big(a(t) - b(t)\big) + Ve(t)}$$

- Make control actions $\alpha(t)$ that **greedily** minimize the bound of the drift-plus-penalty function, $\Delta L(t) + Ve(t)$, in each slot t.
- Advantage: does not require knowledge of the probabilities of the random network events (e.g., task arrivals and channel condition)

From Physical Queues to Virtual Queues

General stochastic optimization problem:

Minimize: $\limsup_{t\to\infty} \overline{y}_0(t)$

Subject to:
1) $\limsup_{t\to\infty} \overline{y}_l(t) \leq 0 \;\; \forall l \in \{1, \ldots, L\}$
2) $\lim_{t\to\infty} \overline{e}_j(t) = 0 \;\; \forall j \in \{1, \ldots, J\}$
3) Queues $Q_k(t)$ are mean rate stable $\forall k \in \{1, \ldots, K\}$
4) $\alpha(t) \in \mathcal{A}_{\omega(t)} \;\; \forall t$

$\bar{y}_l$(t) : the time average expectation of $y_l(t)$ over the first t slots under a particular control strategy

$$\overline{y}_l(t) \triangleq \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\{y_l(\tau)\}$$

Define the time average expectation $\bar{e}_l(t)$ similarly.

How to deal with constraints (1) and (2)?

Define ***virtual queues*** $Z_l(t)$ and $H_j(t)$ with update equations as

$$\begin{aligned} Z_l(t+1) &= \max[Z_l(t) + y_l(t), 0] \\ H_j(t+1) &= H_j(t) + e_j(t) \end{aligned}$$

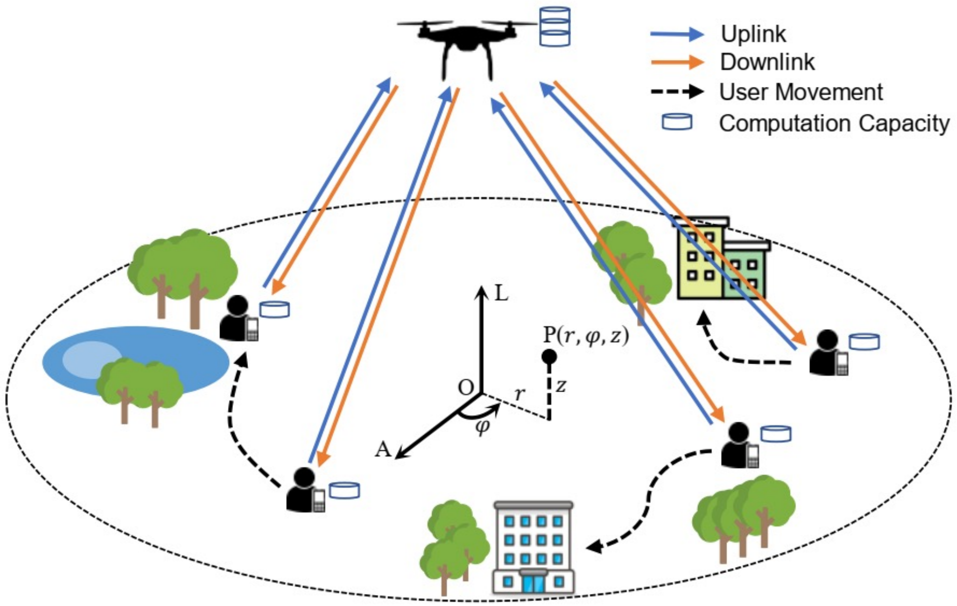
By some mathematical transformations, we obtain

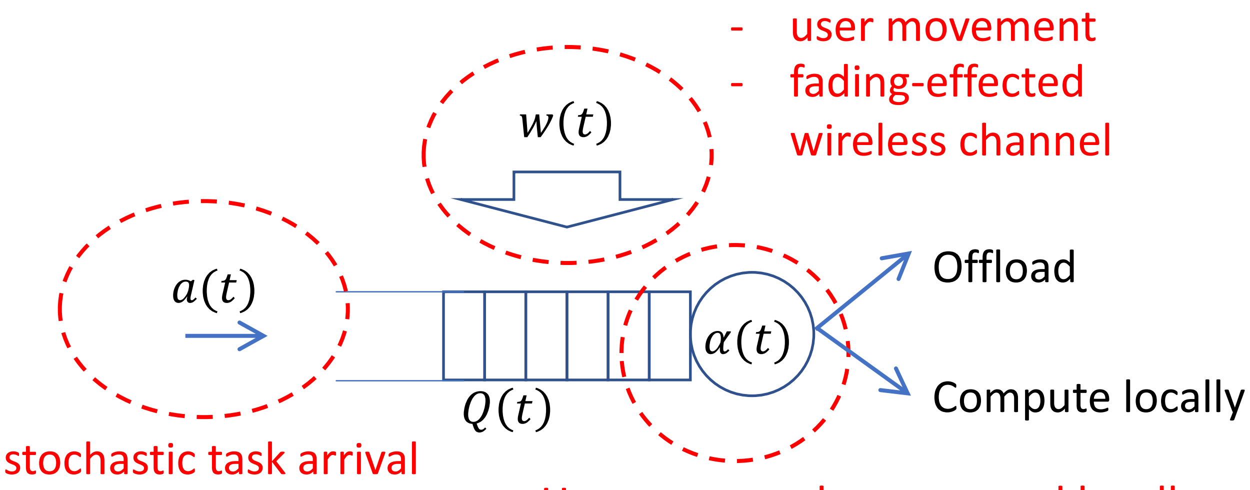
$$\limsup_{t\to\infty} \frac{\mathbb{E}\{Z_l(t)\}}{t} \geq \limsup_{t\to\infty} \overline{y}_l(t)$$

$$\frac{\mathbb{E}\{H_j(t)\} - \mathbb{E}\{H_j(0)\}}{t} = \overline{e}_j(t)$$

Thus, **if $Z_l(t)$ and $H_j(t)$ are mean-rate stable**, i.e., $\lim_{t\to\infty} \frac{E[Z_l(t)]}{t} = 0$ and $\lim_{t\to\infty} \frac{E[H_j(t)]}{t} = 0$, constraints (1) and (2) are satisfied.

Lyapunov-based Optimization in UAV-assisted MEC Networks





System model: a time-evolving network, one UAV serves several mobile users

Objective: minimize system's energy (users and the UAV)

Constraint: stability of users' and the UAV's queues

Lyapunov-based Optimization in UAV-assisted MEC Networks

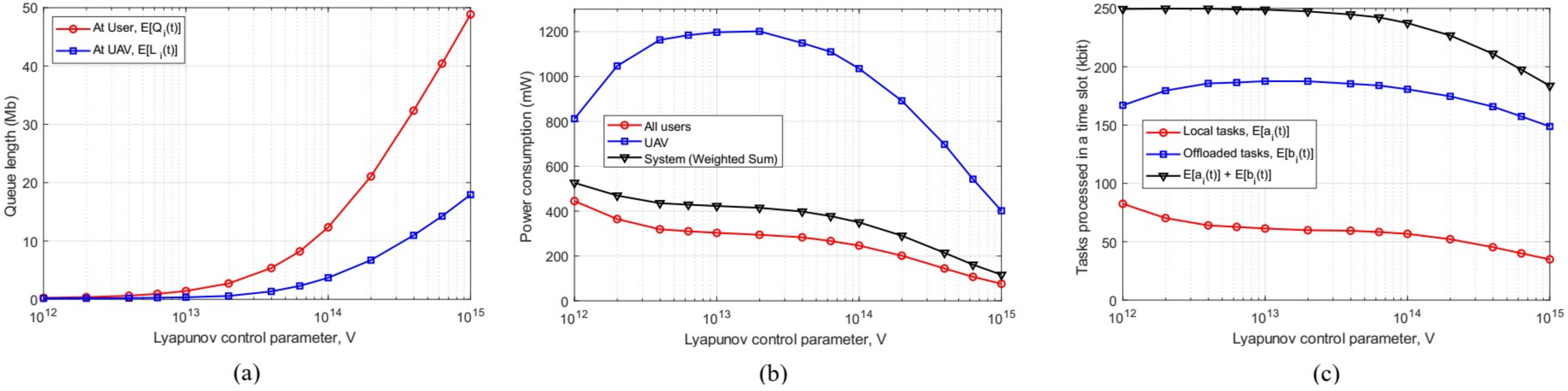


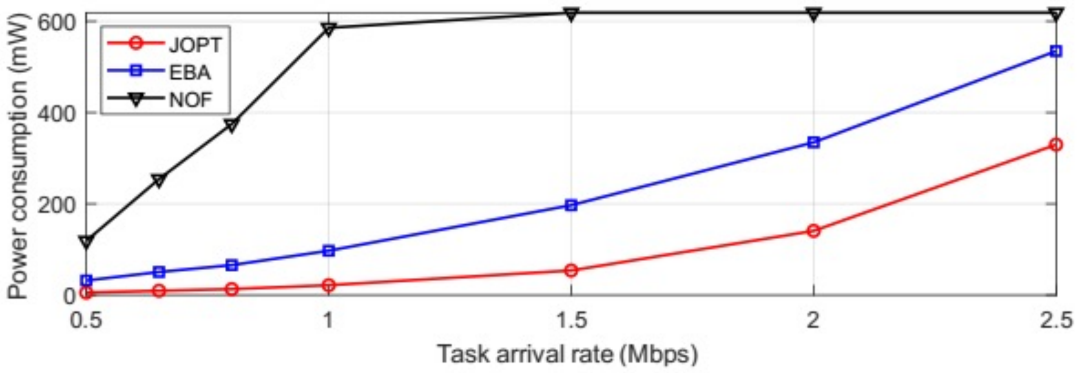
Fig. 3. (a) Average queue length, (b) Power consumption, and (c) Task flow of mobile users with response to the parameter V, $\lambda = 2.5$ Mbps

Increasing V

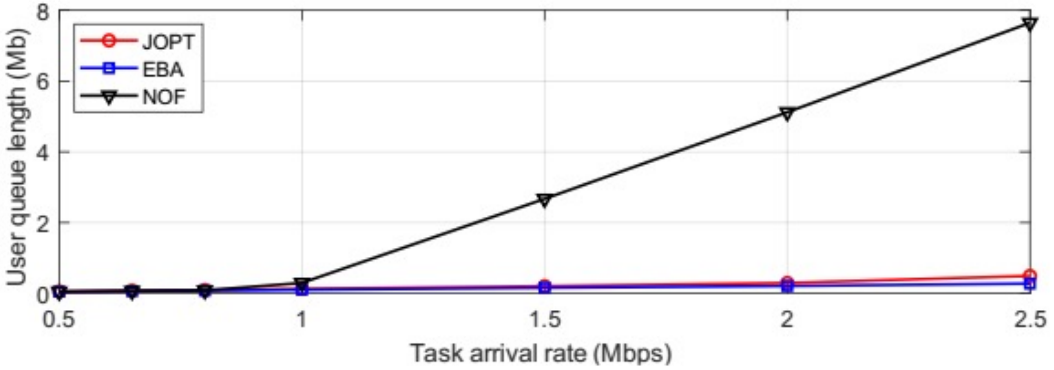
$$\min_{\mathbf{X}(t)} \quad \hat{D}_i(t) + V\left(\psi_1 \sum_{i \in N} E_i(t) + \psi_2 E_{\text{UAV}}(t)\right)$$

- (a) lengthen the task queues at both the user and the UAV
- (b) cause the UAV's power consumption grow rapidly at first and declines gradually afterward
- (c) cause the user upload more tasks to the UAV and process fewer tasks locally -> afterward, process fewer tasks to further save power

Lyapunov-based Optimization in UAV-assisted MEC Networks



(a)



(b)

EBA: Equal Bandwidth Allocation, omitting user movement + time-varying channel condition

JOPT: Joint Optimization for all variables

NOF: No Offloading

Conclusion

- **Lyapunov optimization**: an online optimization approach **for dynamic systems** with **time averages**, including
  - a time-average objective function (e.g., energy consumption),
  - constraints on queue stability,
  - and other constraints in the form of time average
- Algorithm: **greedily** minimize the upper bound of the Lyapunov drift-plus-penalty function in each time slot

References

M. J. Neely, *Stochastic Network Optimization with Application to Communication and Queueing Systems*. Morgan & Claypool, 2010.

Thank you for listening