Reading book seminar
Channel Capacity

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Contents

Fundamentals of information theory

Information

- The “knowledge” we learned about something that we had not been known before
- Example: a sport event
  - Before the event: the results is not known already
  - After the event: the winner is known → giving some new “information”
- The mathematical theory of information is based on probability theory and statistic

Properties of information

- **Property 1:** The more unlikely or unexpected an event is, the more surprising it is → more "information"
  - Example:
    - Vietnam National Football team won AFF Cup
    - Vietnam National Football team will attend World Cup → more information
- **Property 2:** Information is additive

Self-information

- Self-information is a measure of the information content associated with an event in a probability space
- Random variable (RV) X

Sample space

Event	**w_1**	**w_2**	**…**	**w_n**
Probability	$p(w_1)$	$p(w_2)$		$p(w)$

- The self-information of the event w_i is defined as:

$$I(w_i) = log\left(\frac{1}{p(w_i)}\right) = -log\big(p(w_i)\big)$$

$p(w_i) = 1$ → the event is certain to occur → no information

$p(w_i) = 0$ → the event is impossible → infinite information

- Unit: bits (log base 2), nats (log base *e*), Hartley (log base 10)

Entropy

- Average of self-information over the event space
  - Discreate RV

$$H(X) = \sum_i p(w_i) I\,(w_i) = -\sum_i p(w_i) log\big(p(w_i)\big)$$

  - Continuous RV

$$H(X) = \int p(w_i) I(w_i) dw = -\int p(w_i) log\big(p(w_i)\big) dw$$

- Measure the uncertainty of a RV

Entropy (cont.)

- Example: a sport event
  - Equal probability

Event	Team A win	Team B win
Probability	0.5	0.5

$$H(X) = -0.5log(0.5) - 0.5log(0.5) = 1 \text{ (bit)}$$

  - Unequal probability

Event	Team A win	Team B win
Probability	0.2	0.8

$$H(X) = -0.2log(0.2) - 0.8log(0.8) = 0.722 \text{ (bit)}$$

→ The higher the entropy is, the more difficult we can guess the result

Joint Entropy

- Joint entropy of two RVs X and Y

$$H(X,Y) = -\sum_{x,y} p(x,y)\log\big(p(x,y)\big)$$

$p(x,y)$ is the joint probability of X and Y

- Properties
  - $H(X,Y) \geq \max[H(X), H(Y)]$
  - $H(X,Y) \leq \mathrm{H}(X) + H(Y)$, equality happens when X and Y are independent

Conditional Entropy

- Entropy of a RV Y given a RV X is known

$$H(Y|X) = -\sum_{x,y} p(x,y)\log\big(p(x,y)\big) = -\sum_{x,y} p(x,y)\log\left(\frac{p(x,y)}{p(x)}\right)$$

  - The average uncertainty of Y given knowledge about X
  - The uncertainty of Y is reduced ($H(Y|X) \leq H(X)$)

- Properties
  - $H(Y|X) = H(X,Y) - H(X)$
  - If X and Y are independent: $H(Y|X) = H(Y)$

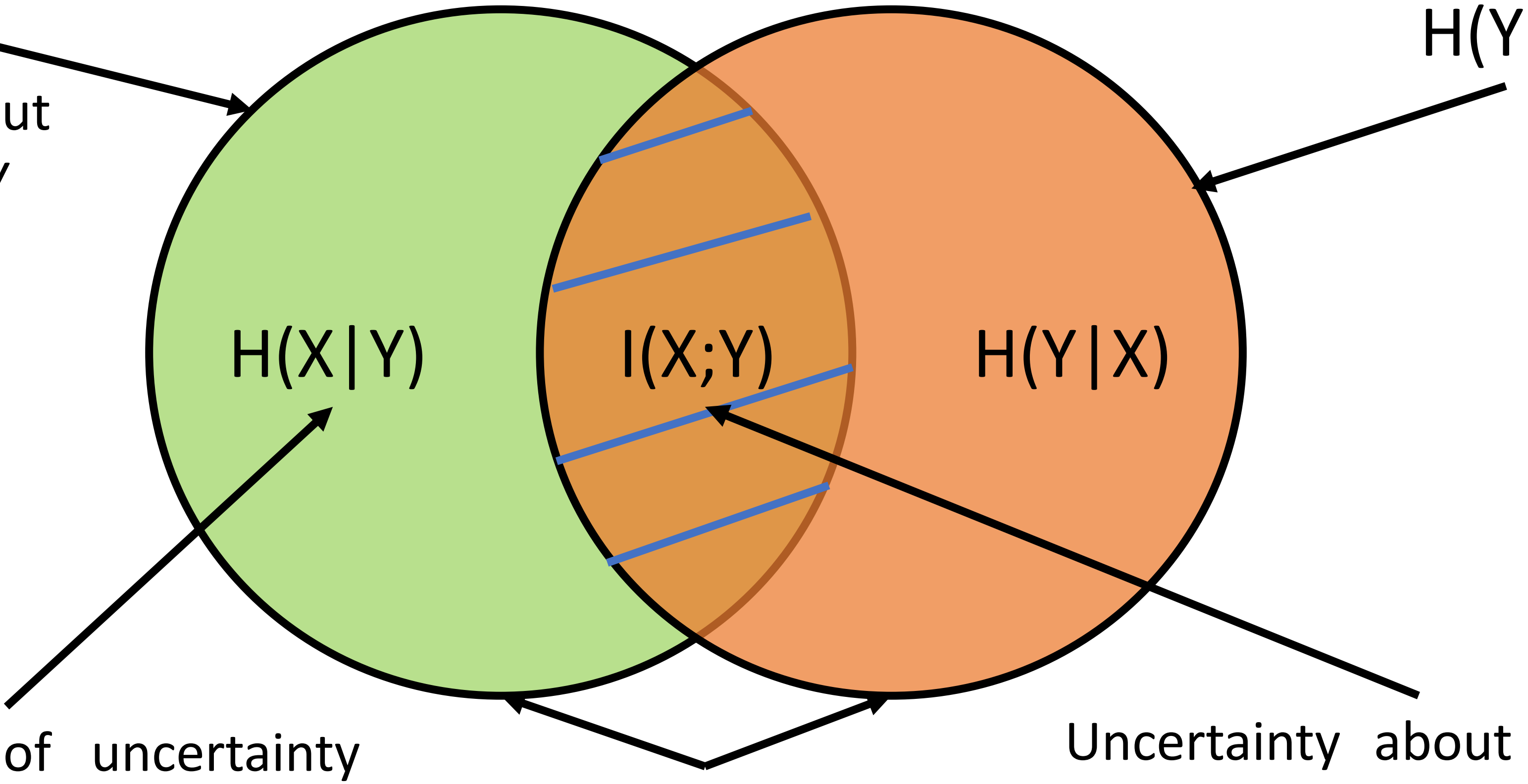
Mutual Information

- The mutual information between X and Y is defined as
$$I(X;Y) = \sum_{x,y} p(x,y) \log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$
  - Measure the information that X and Y share: It measure how much knowing one of these variables reduce uncertainty about the other
- Properties
  - $I(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$
  - X and Y are independent → knowing X does not give any information about Y and vice verse → $I(X,Y) = 0$
  - X = Y → knowing X determines the value of Y and vice verse → $I(X,Y) = H(X) = H(Y)$

Venn diagram

H(X)

The uncertainty about X before observing Y



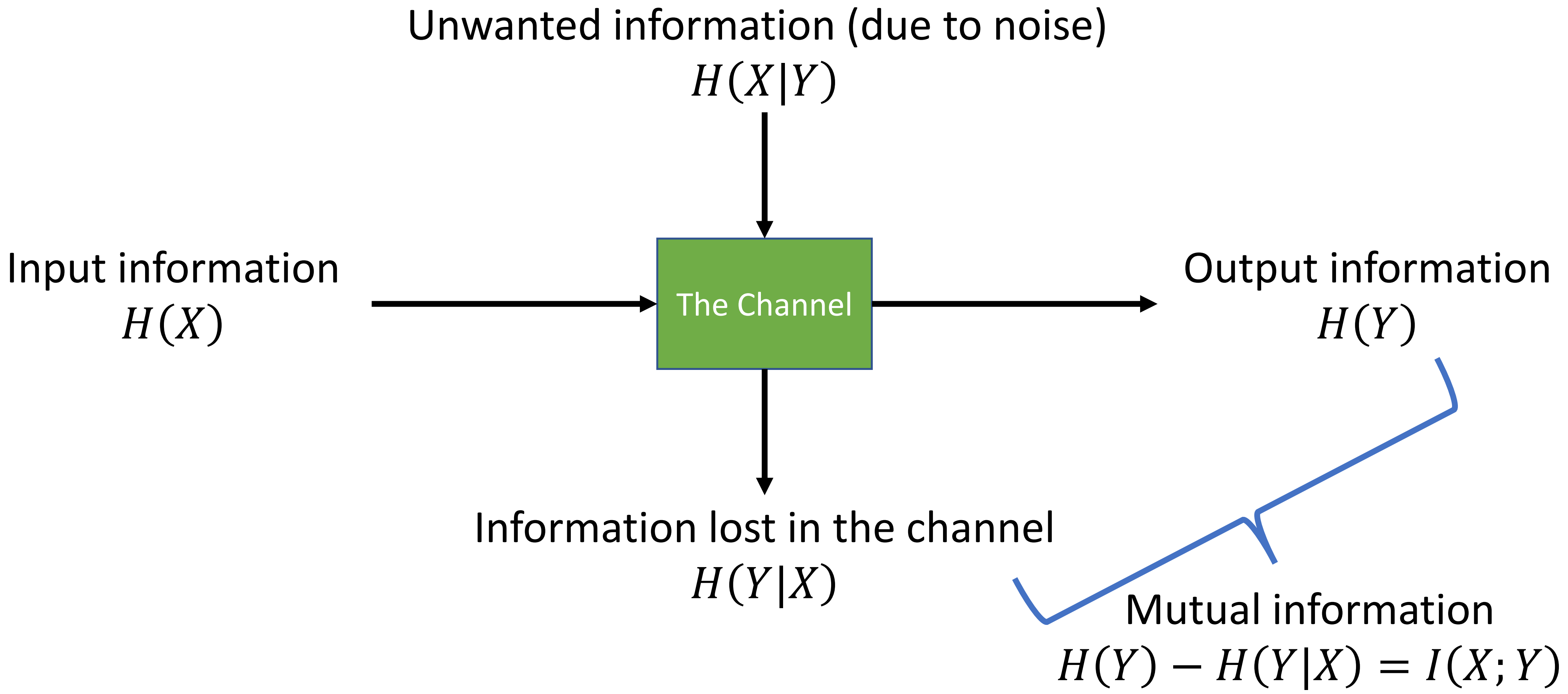
H(Y)

The amount of uncertainty remaining about X after Y has been given

H(X,Y)

Uncertainty about X that is resolved by observing Y

Another interpretation



Capacity of Wireless Channel

Channel Capacity

- Definition: The **maximum information rate** that can be *reliably* transmitted over a communication channel
  - *reliably*: negligible probability of error

Communications over Noisy Channels

X, Y : RVs representing the input and the output of the channel

- Channel capacity is expressed as $C = \max_{p(x)} I(X,Y)$

The mutual information of the channel maximized over all possible input probability distribution

AWGN Channel Capacity

- Information Capacity Theorem (Shannon-Hartley Law):

The information capacity of a continuous channel of bandwidth $W\ Hz$, perturbed by AWGN of power spectral density $N_0/2$ and bandlimited also to $W\ Hz$, is given by

$$C = W\log\left(1 + \frac{P}{N_0W}\right)\text{bits per second}$$

where P is the average transmitted power and $\frac{P}{N_0W}$ is the signal-to-noise ratio or SNR.

Wireless Fading Channel

- **Fading channel:** fluctuation in the transmitted signal power due to the uncertainty of the propagation medium
  - Path loss occurs due to the distance between transmitter and receiver
  - Large-scale fading occurs due to shadowing caused by varying degree of obstructions
  - Small-scale fading occurs due to multipath propagation
- Small-scale fading results in rapid and random fluctuations in the received signal power over a very short period time or very short distance

→ Knowledge of the phenomenon of small-scale fading helps in effective system design

Wireless Fading Channel (cont.)

- **Delay spread ($\boldsymbol{T_\sigma}$)**: different components reach at the receiver at different times in multipath propagation → the received signal spreads over time domain
  - The pulse duration of the transmitted signal T_s
  - The received signal duration $T_s + T_\sigma$ $(T_\sigma \text{ is delay spread})$

- **Coherent Bandwidth ($\boldsymbol{B_c}$)**: indicates the range of frequencies over which a channel is considered to remain static
  - Within the coherent bandwidth: all the frequency components experience the same channel response
  - The coherent bandwidth is derived from delay spread. They are inversely proportional

Wireless Fading Channel (cont.)

- **Doppler Spread ($\boldsymbol{B_\sigma}$)**: A relative change in position of a transmitter of a receiver in dynamic multipath environment → frequency shift → spreading of signal in frequency domain
  - The transmitted signal bandwidth: B_s
  - The received signal bandwidth: $B_s + B_\sigma$ (B_σ is Doppler spread)

- **Coherent Time ($\boldsymbol{T_c}$)**: indicates how often a channel changes its response
  - Within the coherent time: the channel response remain the same
  - The coherent time is derived from Doppler spread. They are inversely proportional

Wireless Fading Channel (Cont.)

- Kinds of small-scale fading:
  - Signal bandwidth B_s, symbol period T_s
  - Coherent bandwidth B_c, coherent time T_c

Small-scale fading

Based on time spreading

Flat fading
$B_s < B_c$

Frequency selective fading
$B_s > B_c$

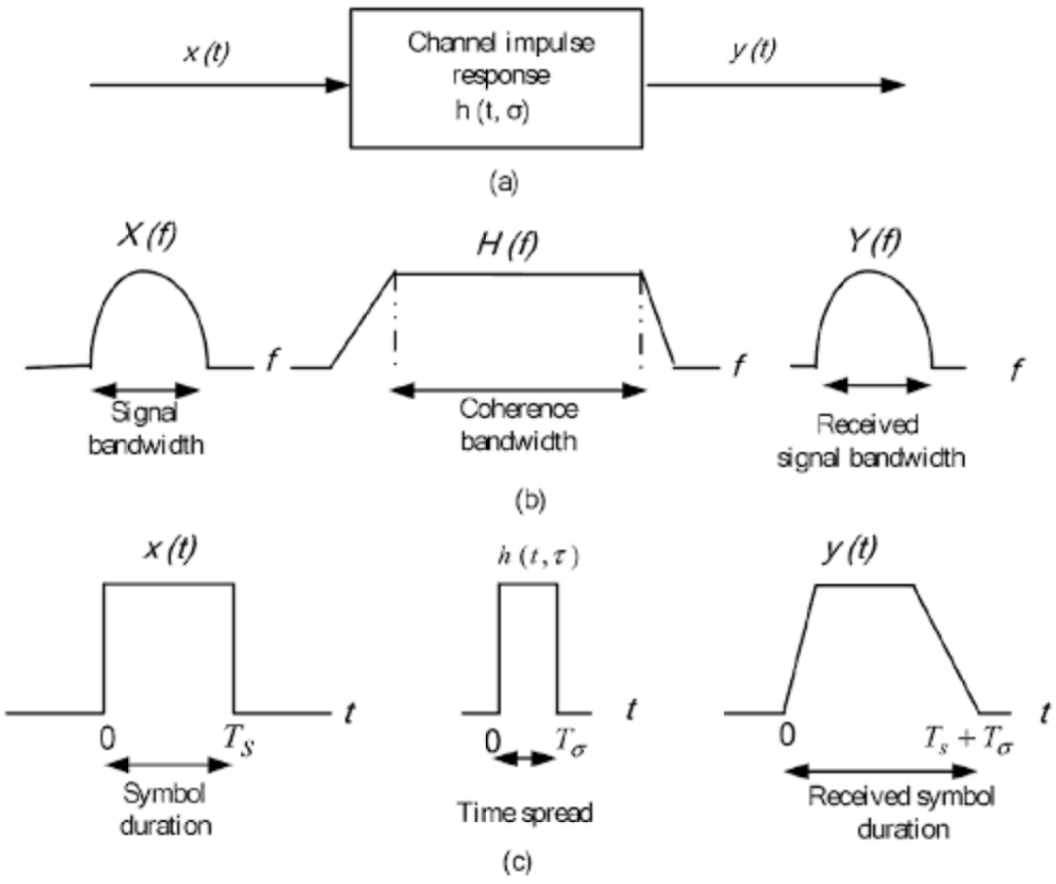
Based on frequency spreading

Fast fading
$T_s > T_c$

Slow fading
$T_s < T_c$

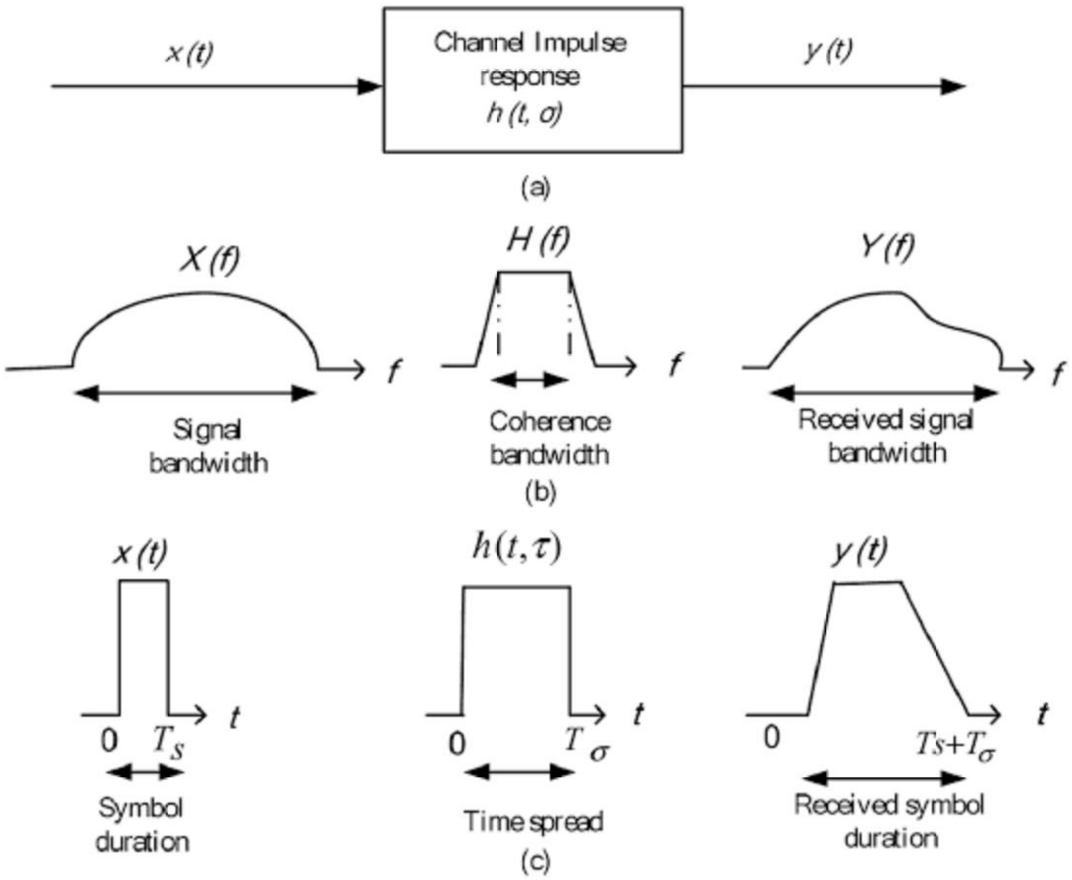
Wireless Fading Channel (Cont.)

- Flat fading: signal propagates through the channel without any distortion (since the amount of channel gain offered across the entire signal bandwidth remains the same)



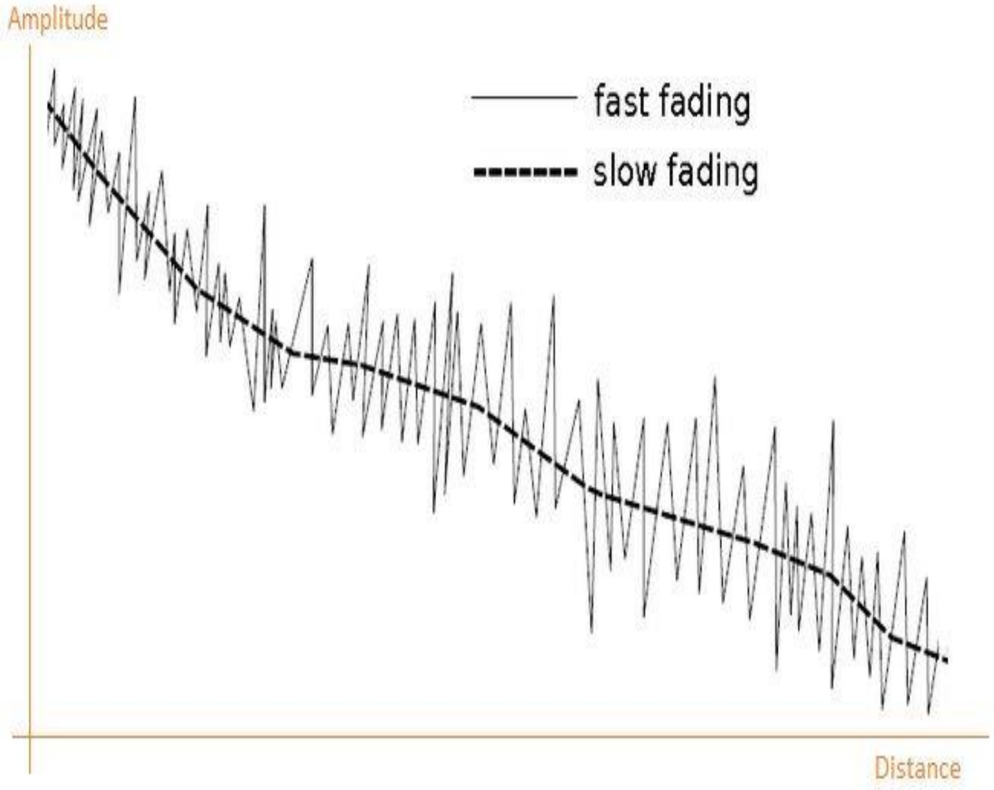
Wireless Fading Channel (Cont.)

- Frequency selective fading: offers constant gain and linear phase response for the bandwidth smaller than the bandwidth of the signal → different parts of the signal bandwidth suffer with different amount of channel gains

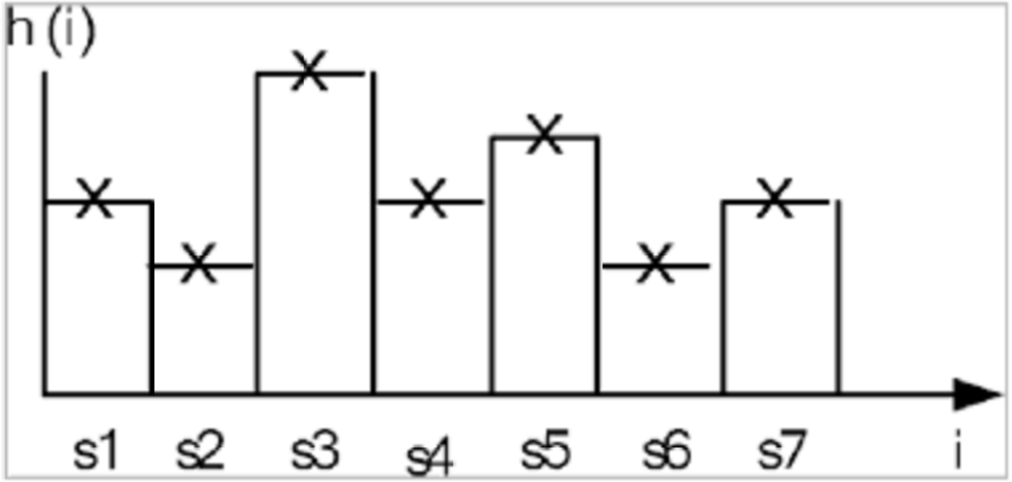


Wireless Fading Channel (Cont.)

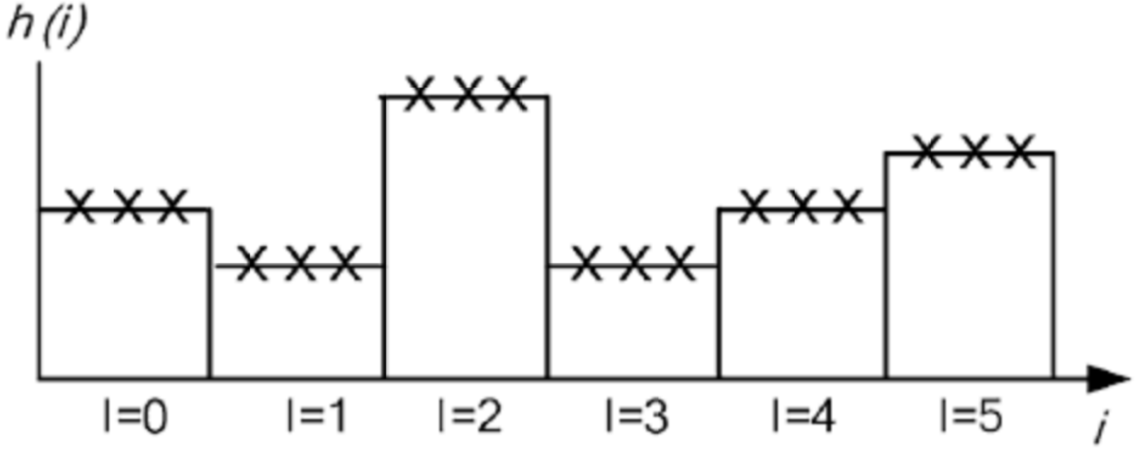
- Fast fading: channel response changes at very short intervals
- Slow fading: channel response changes at long intervals



Wireless Fading Channel (Cont.)



Fast fading



Slow fading

Flat Fading Channel Capacity

- A discrete time flat fading channel can be represented as:
$$Y = hX + N$$
  - h: channel gain, follow a distribution $p(h)$
  - Also known as channel state information (CSI)
- For computation of capacity in flat fading channels, we consider
  - *Channel distribution information (CDI)*: only $p(h)$ is known at transmitter and receiver
  - *Receiver CSI*: h is known at the receiver, both transmitter and receiver know $p(h)$
  - *Transmitter and Receiver CSI*: transmitter and receiver know h and $p(h)$

Known CDI

- Finding the optimal input distribution is generally difficult
  - Depend on the nature of the fading distribution $p(h)$
  - Remain an open problem for almost all $p(h)$

- Solution is available for two specific distribution models:
  - i.i.d Rayleigh channels
  - Finite-state Markov channels

Receiver CSI

- The instant SNR is known to the receiver
- Transmitter can not adapt its transmission strategy relative to the CSI
- Two channel capacity definitions in receiver CSI
  - Shannon (ergodic) capacity
  - Capacity with outage

Receiver CSI (cont.)

- Shannon (ergodic) capacity
  - When fading is fast, the signal is transmitted over the time duration that contain N coherence time period T_c, where in N>>1
  - The transmission is sufficiently long and the fading is fast, all possible channel gain realizations $g[i] = |h[i]|^2$ can take place → the channel capacity can be calculated by averaging out channel capacities for different realizations
  - The corresponding channel capacity related to n-th gain realization

$$C[n] = B\log_2(1 + g[n]\bar{P}/N_0B)$$

Receiver CSI (cont.)

- The Shannon capacity can be obtained by averaging over N coherent time periods as

$$C_N = \frac{1}{N}\sum_{n=1}^{N} B\log_2(1 + g[n]\bar{P}/N_0 B)$$

- Now by letting $N \to \infty$, we obtain

$$C = \mathrm{E}[B\log_2(1+\gamma)] = \int_0^\infty B\log_2(1+\gamma)f(\gamma)d\gamma$$

γ is the instant SNR

- By applying the Jensen's inequality

$$C \le \mathrm{B}\log_2(1+\mathrm{E}[\gamma]) = \mathrm{B}\log_2(1+\bar{\gamma})$$

Receiver CSI (cont.)

- Capacity with outage
  - Problems with Shannon capacity:
    - It assumes an ideal scenario
    - It only provides an upper bound for any practical system
  - Outage capacity is an alternative capacity definition in the presence of CSIR
    - Allows transmitted data to be decoded incorrectly with some probability
    - Applies to slowly-varying channel, where instantaneous SNR is constant over a large number of transmission and then changes to a new value based on the fading distribution

Receiver CSI (cont.)

- Capacity with outage (cont.)
  - Transmitter fixes a minimum SNR γ_{min} for a data rate

$$C = B\log(1 + \gamma_{min})$$

  - If the received SNR is larger than or equal to γ_{min}, the data is correctly received
  - If the received SNR is below than γ_{min}, the bits received can not be decoded correctly → the receiver declares an outage with a probability

$$P_{out} = P(\gamma \leq \gamma_{min})$$

  - Outage capacity

$$C_{out} = (1 - P_{out})B\log_2(1 + \gamma_{min})$$

References

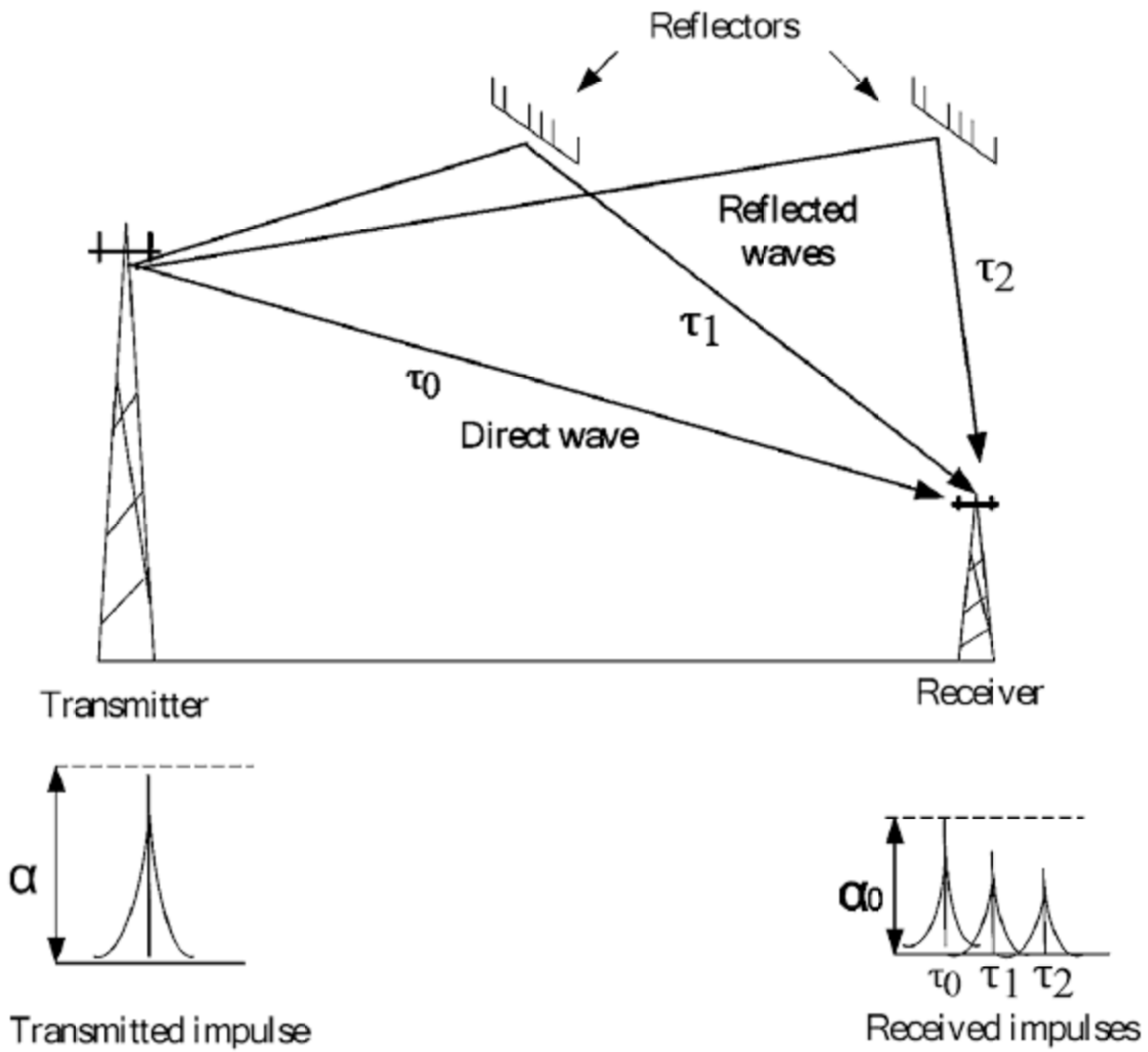
[1] Thanh V. Pham, “Channel Capacity”, Lecture Notes.

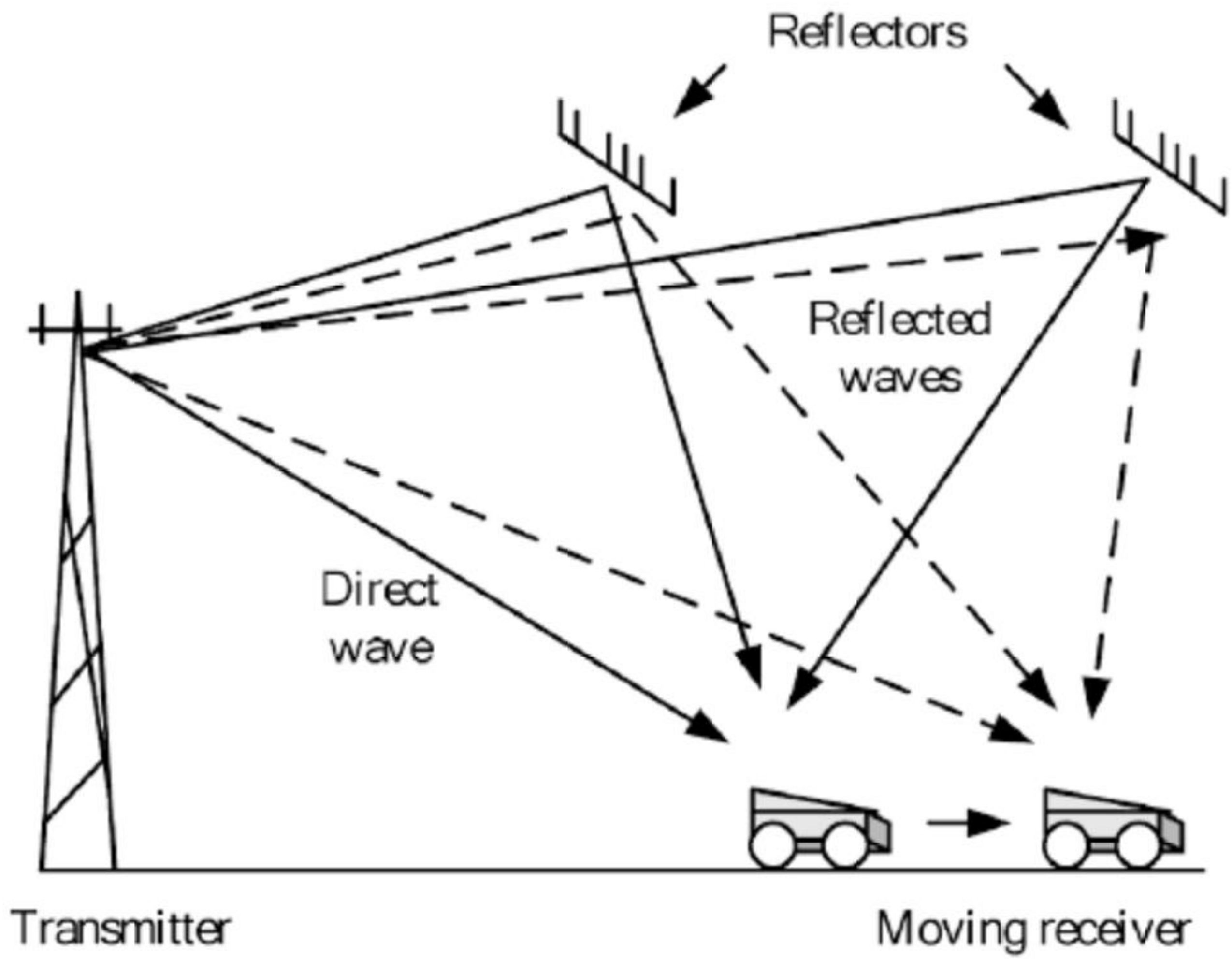
[2] A. Goldsmith, “Wireless Communication”, Cambridge University Press.

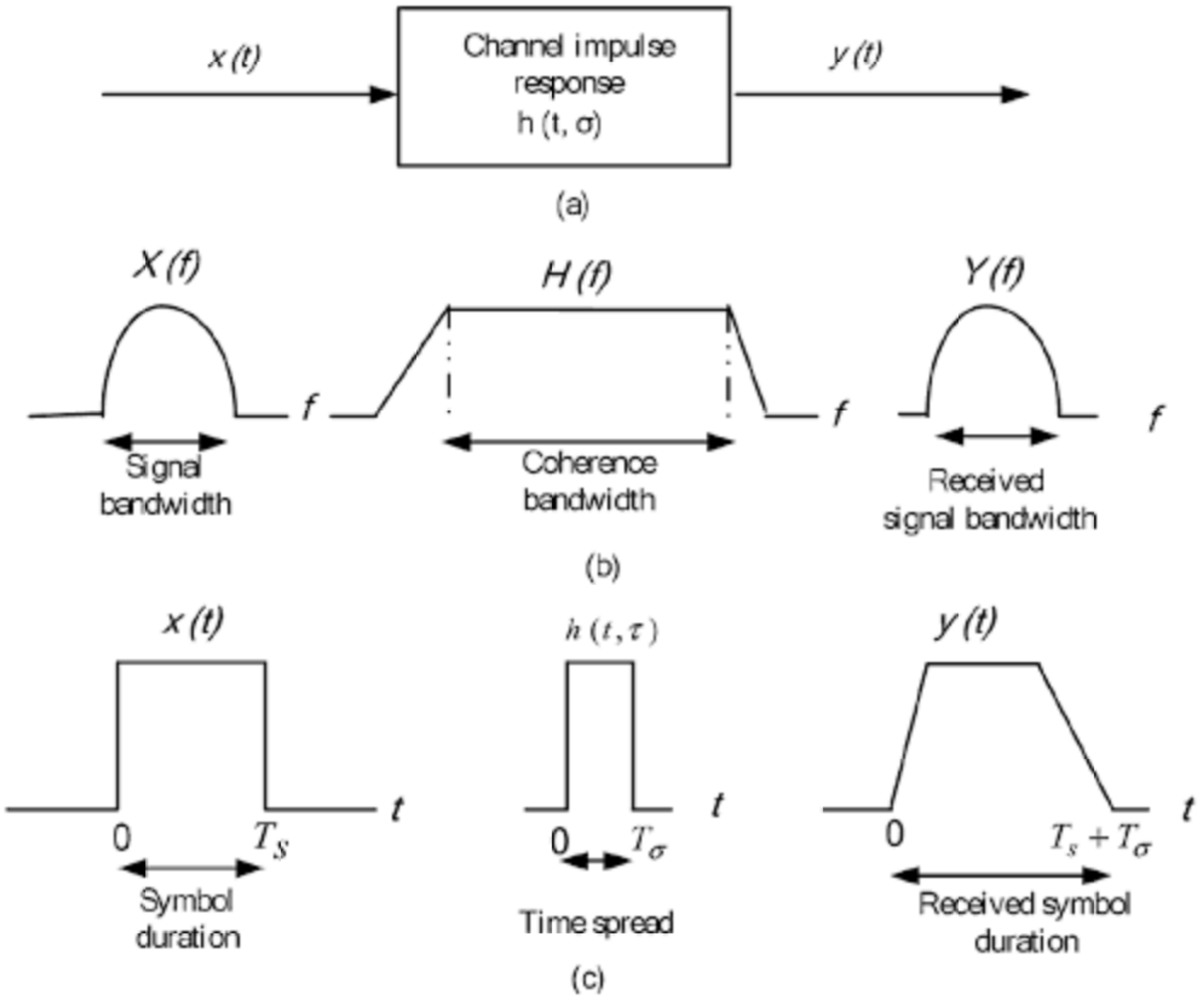
[3] Ivan B. Djordjevic, “Advanced Optical and Wireless Communication Systems”, Springer

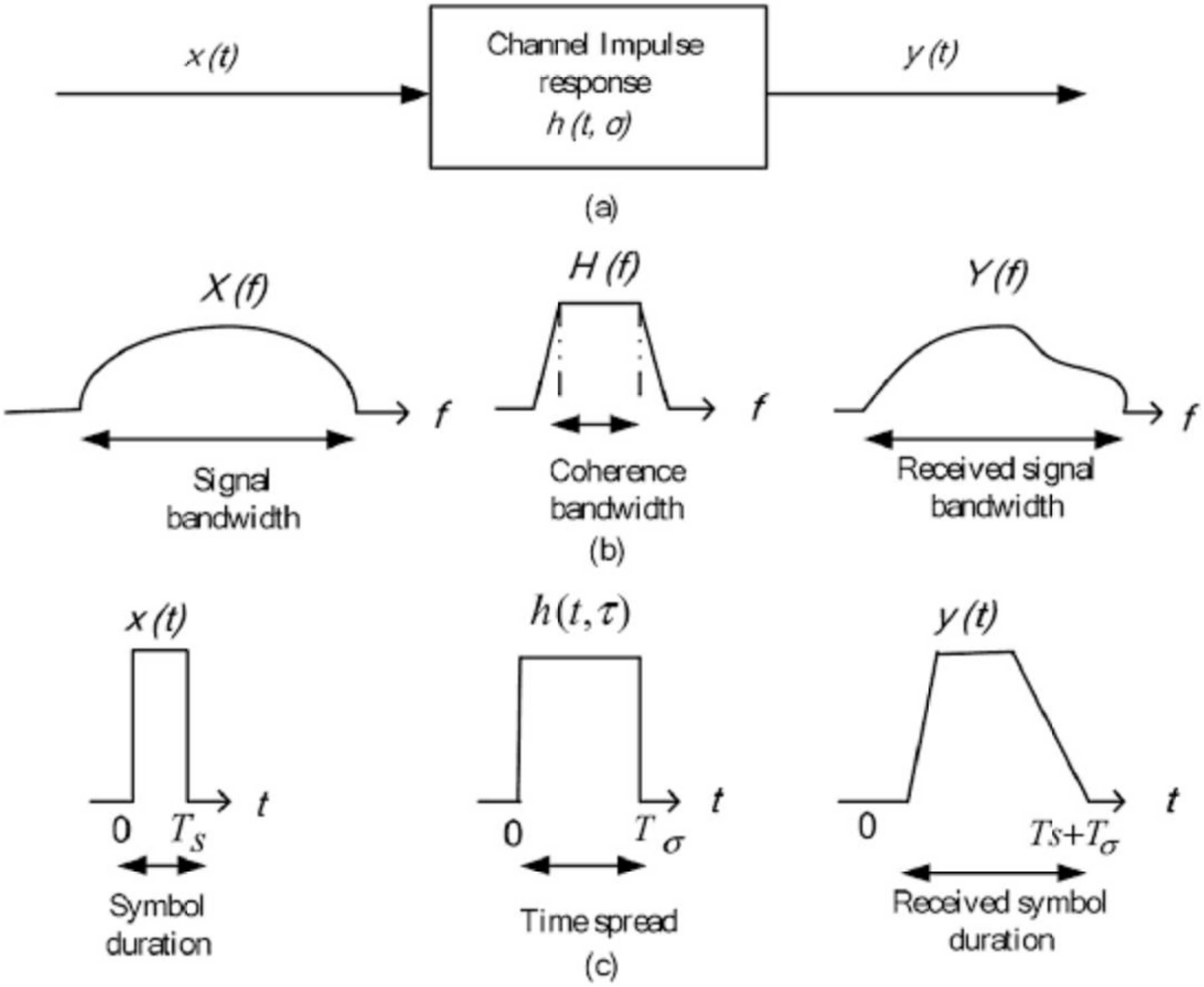
[4] D. Tse, P. Viswanath, “Fundamentals of Wireless Communications”, Cambridge University Press

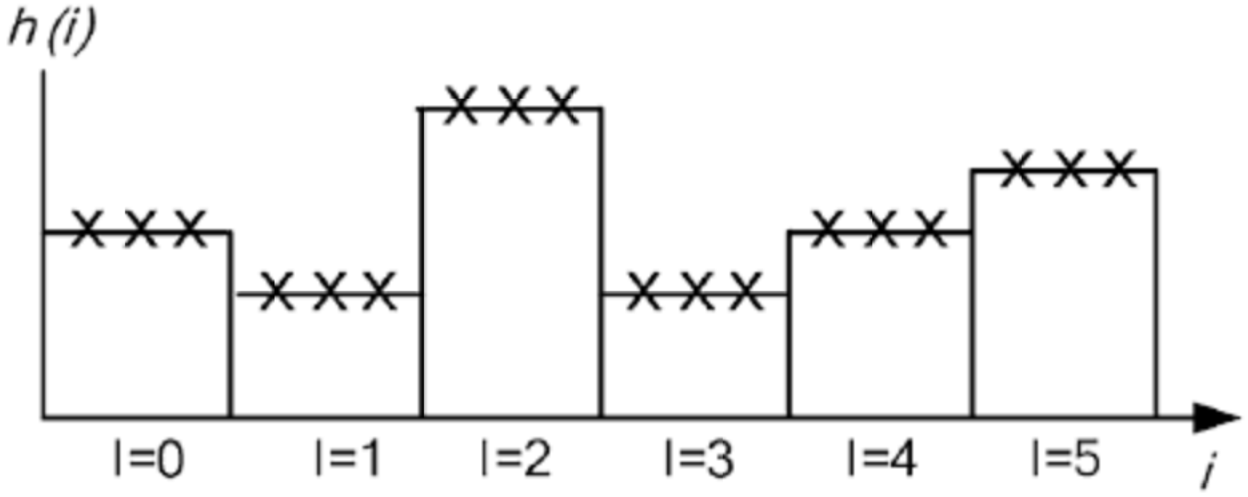
Thank you


Reflectors
Reflected waves
τ_2
τ_1
τ_0
Direct wave
Transmitter
Receiver
α
α_0
τ_0 τ_1 τ_2
Transmitted impulse
Received impulses


Reflectors
Reflected
waves
Direct
wave
Transmitter
Moving receiver


x (t)
Channel impulse response h (t, σ)
y (t)
(a)
X (f)
f
Signal bandwidth
H (f)
f
Coherence bandwidth
Y (f)
f
Received signal bandwidth
(b)
x (t)
t
0
T_S
Symbol duration
$h(t, \tau)$
t
0
T_σ
Time spread
y (t)
t
0
$T_s + T_\sigma$
Received symbol duration
(c)


x (t)
Channel Impulse response
h (t, σ)
y (t)
(a)
X (f)
f
Signal bandwidth
H (f)
f
Coherence bandwidth
Y (f)
f
Received signal bandwidth
(b)
x (t)
0
T_S
t
Symbol duration
$h(t,\tau)$
0
T_σ
t
Time spread
y (t)
0
$Ts+T_\sigma$
t
Received symbol duration
(c)


h (i)
l=0
l=1
l=2
l=3
l=4
l=5
i

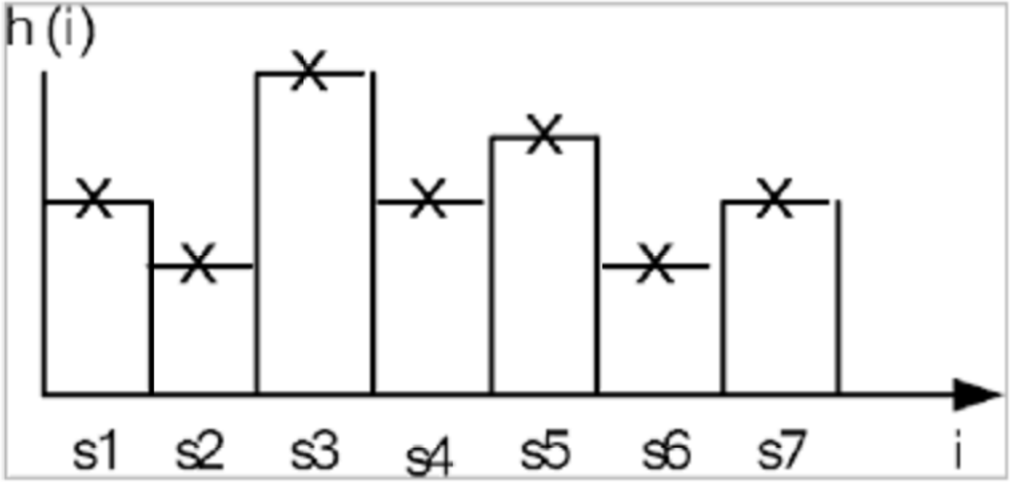
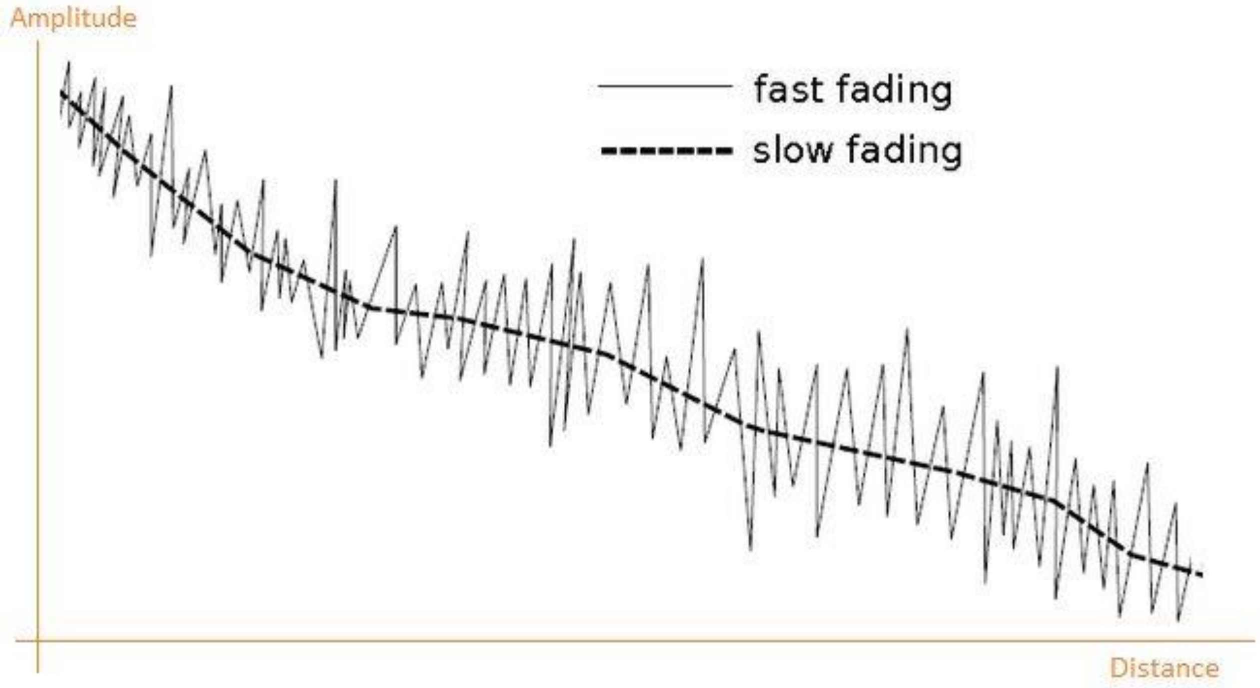


Figure 10.5 A conceptual view of fast fading process


Amplitude
fast fading
slow fading
Distance